Liang General Relativity

Symbols

$\mathscr{F}_M$, $\mathscr{F}$: The set of all C^∞ scalar fields on M. $\mathscr{F}_{R^3}$: The set of all C^∞ functions on $\mathbb{R}^3$. V_p: The set of all vector at $p\in M$.
$\mathscr{T}_V(k,l)$: The set of all tensors of type (k,l) on V. $\mathscr{T}_{V_p}(k,l)$: The set of all tensors of type (k,l) on V_p, $p\in M$.
$\mathscr{F}_M(k,l)$: The set of all C^∞ tensor fields of type (k,l)on M.
$\Lambda_p(l)$: The set all all l-form at $p\in M$. $\Lambda_M(l)$: The set of all l-form field on M.

Manifolds and Vector fields

1. $v:\mathscr{F}_M\to\mathbb{R}$ is a **vector** at $p\in M$, if $\forall f,g\in\mathscr{F}_M$, $\alpha,\beta\in\mathbb{R}$, (a) $v(\alpha f+\beta g)=\alpha v(f)+\beta v(g)$, (b) $v(fg)=f|_p v(g)+g|_p v(f)$.

2. With (O,ψ) and $f\in\mathscr{F}_M$, we have $F(x^1,\cdots,x^n)$. For $p\in O$, we can define n vectors $X_\mu(f)=\left.\dfrac{\partial F(x^1,\cdots,x^n)}{\partial x^\mu}\right|_p$, $\forall f\in\mathscr{F}_M$.

• $\dim V_p=\dim M=n$. $\forall v\in V_p$, $v=v^\mu X_\mu$, where $v^\mu=v(x^\mu)$.

3. For $p\in O$, $\{X_1,\cdots,X_n\}$ is called a **coordinate basis** of V_p. Suppose $p\in O\cap O'$, $v\in V_p$, then $v'^\nu=\left.\dfrac{\partial x'^\nu}{\partial x^\mu}\right|_p v^\mu$.

4. Suppose $C(t)$ is a C^1 curve on M, then the **tangent vector** of $C(t)$ at $C(t_0)$ is defined as $T(f)=\left.\dfrac{\mathrm{d}f(C(t))}{\mathrm{d}t}\right|_{t_0}$, $\forall f\in\mathscr{F}_M$. $T\equiv\left.\dfrac{\partial}{\partial t}\right|_{t_0}$.

• Tangent vector to the x^μ coordinate line $X_\mu(f)=\left.\dfrac{\partial}{\partial x^\mu}\right|_p(f)=\left.\dfrac{\partial f}{\partial x^\mu}\right|_p$. $X_\mu\equiv\dfrac{\partial}{\partial x^\mu}$. For $C(t)$, $\dfrac{\partial}{\partial t}=\dfrac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}\dfrac{\partial}{\partial x^\mu}$.

5. $\left\{X_\mu\equiv\dfrac{\partial}{\partial x^\mu}\right\}$ constitutes n smooth vector fields, called coordinate basis vector fields.

6. $\{e_\mu\}$ is a set of basis vectors of $V\Longrightarrow$ basis vectors of V^* as $e^{\mu*}(e_\nu)=\delta^\mu{}_\nu$. $\forall\omega\in V^*$, $\omega=\omega_\mu e^{\mu*}$, $\omega_\mu\equiv\omega(e_\mu)$. $v=e^{\mu*}(v)e_\mu$.

7. The map $V\to V^{**}$, $v\mapsto v^{**}$ is defined as $v^{**}(\omega)=\omega(v)$, $\forall\omega\in V^*$ and is an isomorphism.

8. $e'_\mu=A^\nu{}_\mu e_\nu$, $e'^{\mu*}=\left(\widetilde{A}^{-1}\right)_\nu{}^\mu e^{\nu*}$.

9. $\left.\mathrm{d}f\right|_p(v)=v(f)$, $\forall v\in V_p$. $\mathrm{d}x^\mu\left(\dfrac{\partial}{\partial x^\nu}\right)=\dfrac{\partial}{\partial x^\nu}\left(x^\mu\right)=\delta^\mu{}_\nu\Rightarrow\{\mathrm{d}x^\mu\}$ are dual coordinate basis vectors. $\omega=\omega_\mu\mathrm{d}x^\mu$, $\omega_\mu=\omega\left(\dfrac{\partial}{\partial x^\mu}\right)$.

• $\mathrm{d}f=\dfrac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu$. $\omega'_\nu=\left.\dfrac{\partial x^\mu}{\partial x'^\nu}\right|_p\omega_\mu$.

10. $\dim\mathscr{T}_V(k,l)=n^{k+l}$. *Example*. $n=2,k=2,l=1$. A basis of $\mathscr{T}_V(2,1)$: $e_1\otimes e_1\otimes e^{1*}$, $e_1\otimes e_1\otimes e^{2*}$, $e_1\otimes e_2\otimes e^{1*}$, $e_1\otimes e_2\otimes e^{2*}$, $\cdots$.

• For $T\in\mathscr{T}_V(2,1)$, $T=T^{\mu\nu}{}_\sigma e_\mu\otimes e_\nu\otimes e^{\sigma*}$, $T^{\mu\nu}{}_\sigma=T(e^{\mu*},e^{\nu*};e_\sigma)$.

• $T\in\mathscr{T}_V(1,1)$, $T:V^*\times V\to\mathbb{R}$ or $T:V\to V$ or $T:V^*\to V^*$. $T=T^\mu{}_\nu e_\mu\otimes e^{\nu*}$, $T^\mu{}_\nu\equiv T(e^{\mu*};e_\nu)$.

11. $T'^\mu{}_\nu=(A^{-1}TA)^\mu{}_\nu$.

Proof. $T'^\mu{}_\nu=T(e'^{\mu*};e'_\nu)=T((\widetilde{A}^{-1})_\rho{}^\mu e^{\rho*};A^\sigma{}_\nu e_\sigma)=(\widetilde{A}^{-1})_\rho{}^\mu A^\sigma{}_\nu T(e^{\rho*};e_\sigma)=(\widetilde{A}^{-1})_\rho{}^\mu A^\sigma{}_\nu T^\rho{}_\sigma=(A^{-1})^\mu{}_\rho T^\rho{}_\sigma A^\sigma{}_\nu=(A^{-1}TA)^\mu{}_\nu$.

12. $T\in\mathscr{T}_V(1,1)$. $CT:=T^\mu{}_\mu$. $T\in\mathscr{T}_V(2,1)$. $C^1_1T=T(e^{\mu*},\bullet;e_\mu)\in V$, $C^1_1T=T(e^{\mu*},e^{\nu*};e_\mu)=T^{\mu\nu}{}_\mu$. $(C^1_1T)^\nu=T(e^{\mu*},e^{\nu*};e_\mu)=T^{\mu\nu}{}_\mu$.

13. $C(v\otimes\omega)=v(\omega)$. $C(v\otimes\omega)=v\otimes\omega(e^{\mu*};e_\mu)=v(e^{\mu*})\omega(e_\mu)=e^{\mu*}(v^\nu e_\nu)\omega(e_\mu)=v^\nu\delta^\mu{}_\nu\omega(e_\mu)=\omega(v)=v(\omega)$.

14. $C^1_2(T\otimes v)=T(\bullet,v)$, $T\in\mathscr{T}_V(0,2)$. $C^2_2(T\otimes\omega)=T(\bullet,\omega;\bullet)$, $T\in\mathscr{T}_V(2,1)$.

Proof. $C^1_2(T\otimes v)=T\otimes v(e^{\mu*};\bullet,e_\mu)=v(e^{\mu*})T(\bullet,e_\mu)=v_\mu T(\bullet,e_\mu)=T(\bullet,v^\mu e_\mu)=T(\bullet,v)$.

15. $T=T^{\mu\nu}{}_\sigma e_\mu\otimes e_\nu\otimes e^{\sigma*}=T^{\mu\nu}{}_\sigma\dfrac{\partial}{\partial x^\mu}\otimes\dfrac{\partial}{\partial x^\nu}\otimes\mathrm{d}x^\sigma$, $T^{\mu\nu}{}_\sigma=T\left(\mathrm{d}x^\mu,\mathrm{d}x^\nu;\dfrac{\partial}{\partial x^\sigma}\right)$. $T'^{\mu\nu}{}_\sigma=\left.\dfrac{\partial x'^\mu}{\partial x^\alpha}\dfrac{\partial x'^\nu}{\partial x^\beta}\dfrac{\partial x^\gamma}{\partial x'^\sigma}\right|_p T^{\alpha\beta}{}_\gamma$.

16. On $\mathbb{R}^2$. $\mathrm{d}l^2=\mathrm{d}x^2+\mathrm{d}y^2=\left[\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right]\mathrm{d}t^2=\left[(T^1)^2+(T^2)^2\right]\mathrm{d}t^2=|T|^2\mathrm{d}t^2\Rightarrow\mathrm{d}l=|T|\mathrm{d}t$, $l=\displaystyle\int|T|\mathrm{d}t$.

17. On (M,g). $g(T,T)=g\left(T^\mu\dfrac{\partial}{\partial x^\mu},T^\nu\dfrac{\partial}{\partial x^\nu}\right)=T^\mu T^\nu g_{\mu\nu}=g_{\mu\nu}\dfrac{\mathrm{d}x^\mu}{\mathrm{d}t}\dfrac{\mathrm{d}x^\nu}{\mathrm{d}t}$. $l=\displaystyle\int\sqrt{|g(T,T)|}\mathrm{d}t=\int\sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}=\int\sqrt{|\mathrm{d}s^2|}$.

• $(\mathbb{R}^n,\delta)$. $\delta=\delta_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu$, $\mathrm{d}s^2\equiv\delta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=\left(\mathrm{d}x^1\right)^2+\left(\mathrm{d}x^2\right)^2+\cdots$.

• $(\mathbb{R}^n,\eta)$. $\eta=\eta_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu$, $\mathrm{d}s^2\equiv\eta_{\mu\nu}\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu=-\left(\mathrm{d}x^0\right)^2+\left(\mathrm{d}x^1\right)^2+\cdots$.

18. $T=T^{\mu\nu}{}_\sigma e_\mu\otimes e_\nu\otimes e^{\sigma*}\Longrightarrow T^{ab}{}_c=T^{\mu\nu}{}_\sigma(e_\mu)^a(e_\nu)^b(e^\sigma)_c$. $T^{\mu\nu}{}_\sigma\equiv T(e^{\mu*},e^{\nu*};e_\sigma)\Longrightarrow T^{\mu\nu}{}_\sigma\equiv T^{ab}{}_c(e^\mu)_a(e^\nu)_b(e_\sigma)^c$.

• $T_{ab}\in\mathscr{T}_V(0,2)$. $T(\bullet,\mu)=C^1_2(T\otimes e_\mu)\equiv T_{ab}(e_\mu)^b\equiv T_{a\mu}$. $T_{a\mu}$ can be considered as the μ-th dual vector in the list $T_{a1},\cdots,T_{an}$.

19. $\delta^a{}_b:V\to V$. $\delta^a{}_bv^b=v^a$, $\delta^a{}_b\omega_a=\omega_b$. $\delta^\mu{}_\nu\equiv\delta^a{}_b(e^\mu)_a(e_\nu)^b\Longrightarrow\delta^a{}_b\equiv\delta^\mu{}_\nu(e_\mu)^a(e^\nu)_b=(e_\mu)^a(e^\mu)_b$.

20. Let $v\in V$, $g(\bullet,v)\in V^*$. $g(\bullet,v)=C^1_2(g\otimes v)=C^1_2(g_{ab}v^c)=g_{ab}v^b\Longrightarrow v_a:=g_{ab}v^b$, $\omega^a:=g^{ab}\omega_b$.

• $g_{ac}T^c{}_b=g(\bullet,e_\mu)\otimes T(e^{\mu*};\bullet)\Longrightarrow T_{ab}:=g_{ac}T^c{}_b$.

• $g_{\mu\nu}v^\nu=g_{ab}(e_\mu)^a(e_\nu)^bv^c(e^\nu)_c=g_{ab}v^c(e_\mu)^a(e_\nu)^b(e^\nu)_c=g_{ab}v^c(e_\mu)^a\delta^b{}_c=g_{ab}v^b(e_\mu)^a=v_a(e_\mu)^a=v_\mu$.

• $\omega^a=g^{ab}\omega_b=g^{ab}(g_{bc}\omega^c)\Longrightarrow g^{ca}g_{ab}=\delta^c{}_b$, $g^{\mu\nu}g_{\nu\sigma}=\delta^\mu{}_\sigma$.

• $g_{ab}\left(\frac{\partial}{\partial x^\mu}\right)^b = g_{\mu\nu}(\mathrm{d}x^\nu)_a$, $\eta_{ab}\left(\frac{\partial}{\partial x^0}\right)^b = -(\mathrm{d}x^0)_a$. $g^{ab}(\mathrm{d}x^\mu)_b = g^{\mu\nu}\left(\frac{\partial}{\partial x^\nu}\right)^a$.

21. $\eta_{ab} := \eta_{\mu\nu}(\mathrm{d}x^\mu)_a(\mathrm{d}x^\nu)_b = -(\mathrm{d}t)_a(\mathrm{d}t)_b + (\mathrm{d}x)_a(\mathrm{d}x)_b + (\mathrm{d}y)_a(\mathrm{d}y)_b + (\mathrm{d}z)_a(\mathrm{d}z)_b$.

• $\{t,r,\theta,\phi\}$: $\eta_{ab} = -(\mathrm{d}t)_a(\mathrm{d}t)_b + (\mathrm{d}r)_a(\mathrm{d}r)_b + r^2(\mathrm{d}\theta)_a(\mathrm{d}\theta)_b + r^2\sin^2\theta(\mathrm{d}\phi)_a(\mathrm{d}\phi)_b$. $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$.

22. $T_{(ab)} := \frac{1}{2}(T_{ab}+T_{ba})$, $T_{[ab]} := \frac{1}{2}(T_{ab}-T_{ba})$. $T_{a_1\cdots a_l} = T_{(a_1\cdots a_l)} \Rightarrow T_{a_1\cdots a_l} = T_{a_{\pi(1)}\cdots a_{\pi(l)}}$. $T_{a_1\cdots a_l} = T_{[a_1\cdots a_l]} \Rightarrow T_{a_1\cdots a_l} = \delta_\pi T_{a_{\pi(1)}\cdots a_{\pi(l)}}$.

• $T_{[a_1\cdots a_l]}S^{a_1\cdots a_l} = T_{[a_1\cdots a_l]}S^{[a_1\cdots a_l]} = T_{a_1\cdots a_l}S^{[a_1\cdots a_l]}$. $T_{[[ab]c]} = T_{[abc]}$, where $T_{[[ab]c]} = \frac{1}{2}(T_{[abc]} - T_{[bac]})$.

• $T_{[(ab)c]} = 0$, $T_{(a[bcd])} = 0$. $T^{(abc)}S_{[abc]} = 0$. $T_{a_1\cdots a_l} = T_{[a_1\cdots a_l]} \Rightarrow T_{(a_1\cdots a_l)} = 0$, $T_{a_1\cdots a_l} = T_{(a_1\cdots a_l)} \Rightarrow T_{[a_1\cdots a_l]} = 0$.

Riemann curvature

1. $\nabla_a : \mathscr{F}_M(k,l) \to \mathscr{F}_M(k,l+1)$ is a derivative operator if (a) linearity. (b) Leibnitz. (c) $v(f) = v^a\nabla_a f$. (d) $\nabla_a\nabla_b f = \nabla_b\nabla_a f$.

• (e) Commutativity with contraction. $\nabla_a(v^b\omega_b) = v^b\nabla_a\omega_b + \omega_b\nabla_a v^b$.

• $\nabla_a f = (\mathrm{d}f)_a$. *Proof.* $(\mathrm{d}f)_a v^a = v(f) = v^a\nabla_a f$. $\nabla_a f = \tilde{\nabla}_a f = (\mathrm{d}f)_a$.

2. $\omega'_b\big|_p = \omega_b\big|_p \Rightarrow [(\tilde{\nabla}_a - \nabla_a)\omega_b]_p = [(\tilde{\nabla}_a - \nabla_a)\omega'_b]_p$. $\tilde{\nabla}_a - \nabla_a : \mathscr{T}_{V_p}(0,1) \to \mathscr{T}_{V_p}(0,2)$. $\tilde{\nabla}_a - \nabla_a = C^c{}_{ab}$.

• $\nabla_a\omega_b = \tilde{\nabla}_a\omega_b - C^c{}_{ab}\omega_c$. $\nabla_a v^b = \tilde{\nabla}_a v^b + C^b{}_{ac}v^c$. $\nabla_a T^b{}_c = \tilde{\nabla}_a T^b{}_c + C^b{}_{ad}T^d{}_c - C^d{}_{ac}T^b{}_d$.

Proof. $\nabla_a(\omega_b v^b) = \omega_b\nabla_a v^b + v^b\nabla_a\omega_b = \omega_b\nabla_a v^b + v^b(\tilde{\nabla}_a\omega_b - C^c{}_{ab}\omega_c)$. $\tilde{\nabla}_a(\omega_b v^b) = \nabla_a(\omega_b v^b) = \omega_b\tilde{\nabla}_a v^b + v^b\tilde{\nabla}_a\omega_b$

4. $\partial_a T^b{}_c = (\mathrm{d}x^\mu)_a\left(\frac{\partial}{\partial x^\nu}\right)^b(\mathrm{d}x^\sigma)_c\partial_\mu T^\nu{}_\sigma$. $\partial_a v^b = (\mathrm{d}x^\mu)_a\left(\frac{\partial}{\partial x^\nu}\right)^b v^\nu{}_{,\mu}$, $v^\nu{}_{,\mu} \equiv \partial_\mu v^\nu \equiv \frac{\partial v^\nu}{\partial x^\mu}$. $\nabla_a v^b = (\mathrm{d}x^\mu)_a\left(\frac{\partial}{\partial x^\nu}\right)^b v^\nu{}_{;\mu}$.

• $\partial_a\left(\frac{\partial}{\partial x^\nu}\right)^b = 0$, $\partial_a(\mathrm{d}x^\nu)_b = 0$. $\partial_a\partial_b T = \partial_b\partial_a T$, $\partial_{[a}\partial_{b]}T = 0$.

• $v^\nu{}_{;\mu} = v^\nu{}_{,\mu} + \Gamma^\nu{}_{\mu\sigma}v^\sigma$, $\omega_{\nu;\mu} = \omega_{\nu,\mu} - \Gamma^\sigma{}_{\mu\nu}\omega_\sigma$.

5. $[u,v]^a = u^b\nabla_b v^a - v^b\nabla_b u^a$. $[u,v]^\mu = (\mathrm{d}x^\mu)_a[u,v]^a = u^\nu\partial_\nu v^\mu - v^\nu\partial_\nu u^\mu$.

6. A vector field v^a on $C(t)$ is called **parallelly transported along** $C(t)$ if $T^b\nabla_b v^a = 0$.

7. $T^b\nabla_b v^a = \left(\frac{\partial}{\partial x^\mu}\right)^a\left(\frac{\mathrm{d}v^\mu}{\mathrm{d}t} + \Gamma^\mu{}_{\nu\sigma}T^\nu v^\sigma\right)$. $T^b\nabla_b v^a = 0 \Longleftrightarrow \frac{\mathrm{d}v^\mu(t)}{\mathrm{d}t} + \Gamma^\mu{}_{\nu\sigma}T^\nu v^\sigma = 0$.

8. Consider (M, g_{ab}). $u^a v_a \equiv g_{ab}u^a v^b$ is constant $\Longleftrightarrow \nabla_c g_{ab} = 0$. For (M, g_{ab}) there is a unique ∇_a s.t. $\nabla_a g_{bc} = 0$.

Proof. $0 = \nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d{}_{ab}g_{dc} - C^d{}_{ac}g_{bd} = \tilde{\nabla}_a g_{bc} - C_{cab} - C_{bac} \Rightarrow C_{cab} + C_{bac} = \tilde{\nabla}_a g_{bc} \Rightarrow C_{cab} = \frac{1}{2}(\tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab})$

$\Longrightarrow C^c{}_{ab} = g^{cd}C_{dab}$. $\Gamma^c{}_{ab} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$. $\Gamma^\sigma{}_{\mu\nu} = \Gamma^c{}_{ab}(\mathrm{d}x^\sigma)_c\left(\frac{\partial}{\partial x^\nu}\right)^a\left(\frac{\partial}{\partial x^\nu}\right)^b$.

$\Longrightarrow \Gamma^\sigma{}_{\mu\nu} = \frac{1}{2}(\mathrm{d}x^\sigma)_c\left(\frac{\partial}{\partial x^\mu}\right)^a\left(\frac{\partial}{\partial x^\nu}\right)^b g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\rho g_{\mu\rho} - \partial_\rho g_{\mu\nu}) = \frac{1}{2}g^{\sigma\rho}(g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho})$.

9. A **geodesic** on (M, ∇_a) satisfies $T^b\nabla_b T^a = 0$. In a coordinate system, $\frac{\mathrm{d}T^\mu}{\mathrm{d}t} + \Gamma^\mu{}_{\nu\sigma}T^\nu T^\sigma = 0$, $\frac{\mathrm{d}^2x^\mu}{\mathrm{d}t^2} + \Gamma^\mu{}_{\nu\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\sigma}{\mathrm{d}t} = 0$.

• Geodesic $\xrightarrow{\text{re-parameterized}}$ $T'^b\nabla_b T'^a = \alpha T'^a$.

10. $p \in M$ and a vector v^a at p uniquely determines the geodesic $\gamma(t)$ that satisfies (a) $\gamma(0) = p$. (b) Its tangent vector at p is v^a.

11. $f \in \mathscr{F}_M$, $\omega_a \in \mathscr{F}(0,1)$, then $(\nabla_a\nabla_b - \nabla_b\nabla_a)(f\omega_c) = f(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c$.

12. $\omega_c, \omega'_c \in \mathscr{F}(0,1)$ and $\omega'_c\big|_p = \omega_c\big|_p$, then $[(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega'_c]_p = [(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c]_p \Rightarrow (\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c = R_{abc}{}^d\omega_d$.

• For $(\mathbb{R}^n, \delta_{ab}/\eta_{ab})$, $(\partial_a\partial_b - \partial_b\partial_a)\omega_c = (\mathrm{d}x^\mu)_a(\mathrm{d}x^\nu)_b(\mathrm{d}x^\sigma)_c(\partial_\mu\partial_\nu\omega_\sigma - \partial_\nu\partial_\mu\omega_\sigma) = 0$.

13. $(\nabla_a\nabla_b - \nabla_b\nabla_a)v^c = -R_{abd}{}^c v^d$. $(\nabla_a\nabla_b - \nabla_b\nabla_a)T^c{}_d = R_{abd}{}^e T^c{}_e - R_{abe}{}^c T^e{}_d$.

14. $(M, g_{ab} \to \nabla_a) : R_{abcd} \equiv g_{ed}R_{abc}{}^e$. (1) $R_{abc}{}^d = -R_{bac}{}^d$. (2) $R_{[abc]}{}^d = 0$. (3) $\nabla_{[a}R_{bc]d}{}^e = 0$. (4) $R_{abcd} = -R_{abdc}$. (5) $R_{abcd} = R_{cdab}$.

Proof. $R_{[abc]}{}^d\omega_d = \nabla_{[a}\nabla_b\omega_{c]} - \nabla_{[b}\nabla_a\omega_{c]} = 2\nabla_{[a}\nabla_b\omega_{c]}$. $\nabla_a(\nabla_b\omega_c) = \partial_a\nabla_b\omega_c - \Gamma^d{}_{ab}\nabla_d\omega_c - \Gamma^d{}_{ac}\nabla_d\omega_c = \partial_a(\partial_b\omega_c - \Gamma^e{}_{bc}\omega_e) - \Gamma^d{}_{ab}\nabla_d\omega_c - \Gamma^d{}_{ac}\nabla_d\omega_c \Longrightarrow \nabla_{[a}(\nabla_b\omega_{c]}) = \partial_{[a}\partial_b\omega_{c]} - \Gamma^e{}_{[bc}\partial_{a]}\omega_e - \omega_e\partial_{[a}\Gamma^e{}_{bc]} - \Gamma^d{}_{[ab}\nabla_{|d|}\omega_{c]} - \Gamma^d{}_{[ac}\nabla_{b]}\omega_d = 0$

15. **Ricci tensor**. $R_{ac} \equiv g^{bd}R_{abcd}$ or $R_{ac} \equiv R_{abc}{}^b$ (without metric). Scalar curvature $R \equiv g^{ac}R_{ac}$.

16. **Weyl tensor**. $C_{abcd} := R_{abcd} - \frac{2}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{2}{(n-1)(n-2)}Rg_{a[c}g_{d]b}$.

17. $R_{\mu\nu\sigma}{}^\rho = \Gamma^\rho{}_{\mu\sigma,\nu} - \Gamma^\rho{}_{\nu\sigma,\mu} + \Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\nu\lambda} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\rho{}_{\mu\lambda}$.

Lie derivative, Killing vector field, hypersurface

1. Pullback $\phi^* : \mathscr{F}_N \to \mathscr{F}_M$, $f \mapsto \phi^* f$ is defined as $(\phi^* f)\big|_p := f\big|_{\phi(p)}$, $\forall f \in \mathscr{F}_N$, $\forall p \in M$.

2. Pushforward $\phi_* : V_p \to V_{\phi(p)}$, $v^a \mapsto \phi_* v^a$ is defined as $(\phi_* v)(f) := v(\phi^* f)$, $\forall p \in M, \forall v^a \in V_p, \forall f \in \mathscr{F}_N$.

3. Pullback $\phi^* : \mathscr{F}_N(0,l) \to \mathscr{F}_M(0,l)$, $(\phi^* T)_{a_1\cdots a_l}\big|_p(v_1)^{a_1}\cdots(v_l)^{a_l} := T_{a_1\cdots a_l}\big|_{\phi(p)}(\phi_* v_1)^{a_1}\cdots(\phi_* v_l)^{a_l}$, $\forall T \in \mathscr{F}_N(0,l), p \in M, v_1,.., v_l \in V_p$

4. Pushforward $\phi_* : \mathscr{T}_{V_p}(k,0) \to \mathscr{T}_{V_{\phi(p)}}(k,0)$, $(\phi_* T)^{a_1\cdots a_k}(\omega^1)_{a_1}\cdots(\omega^k)_{a_k} := T^{a_1\cdots a_k}(\phi^*\omega^1)_{a_1}\cdots(\phi^*\omega^k)_{a_k}$, $\forall p \in M, T \in \mathscr{T}_{V_p}(k,0), \forall\omega^1,\cdots,\omega^k \in V^*_{\phi(p)}$, where $(\phi^*\omega)_a v^a := \omega_a(\phi_* v)^a$, $\forall v^a \in V_p$.

5. $\phi : M \to N$ is diffeomorphism. $\phi_* : \mathscr{F}_M(k,l) \to \mathscr{F}_N(k,l)$, $(\phi_* T)^a{}_b\big|_q w_a v^b := T^a{}_b\big|_{\phi^{-1}(q)}(\phi^*\omega)_a(\phi^* v)^b$, where $(\phi^* v)^b \equiv (\phi_*^{-1}v)^b$.

6. $C(t)$ is a curve in M, T^a is the tangent vector at $C(t_0)$, then $\phi_* T^a \in V_{\phi(C(t_0))}$ is the tangent vector at $\phi(C(t_0))$ of curve $\phi(C(t))$.
7. $p: M \to N$ is diffeomorphism. $p \in O_1$, $\phi(p) \in O_2$. $\forall q \in \phi^{-1}[O_2], x'^\mu(q) := y^\mu(\phi(q))$.
- $\{x'^\mu\}$ coordinate line $\xrightarrow{\phi}$ $\{y^\mu\}$ coordinate line. $\phi_*\left[\left(\frac{\partial}{\partial x'^\mu}\right)^a\Big|_p\right] = \left(\frac{\partial}{\partial y^\mu}\right)^a\Big|_{\phi(p)}$, $\quad \phi_*\left[(\mathrm{d}x'^\mu)_a\Big|_p\right] = (\mathrm{d}y^\mu)^a\Big|_{\phi(p)}$.
- $(\phi_* T)^\mu{}_\nu\big|_{\phi(p)} = T'^\mu{}_\nu\big|_p$.

8. **Lie derivative** $\mathscr{L}_v T^a{}_b := \lim_{t\to 0}\frac{1}{t}\left(\phi_t^* T^a{}_b - T^a{}_b\right)$.
9. $\mathscr{L}_v f = v(f)$, $\forall f \in \mathscr{F}$. *Proof.* $\mathscr{L}_v f\Big|_p \equiv \lim_{t\to 0}\frac{1}{t}(\phi_t^* f - f)\Big|_p = \lim_{t\to 0}\frac{1}{t}\left[f\Big|_{\phi(p)} - f\Big|_p\right] = \lim_{t\to 0}\frac{1}{t}[f(C(t) - f(C(0))] = \frac{\mathrm{d}}{\mathrm{d}t}f(C(t))\big|_{t=0} \equiv v(f)$.
10. In the adapted coordinate system of v^a, $(\mathscr{L}_v T)^\mu{}_\nu = \dfrac{\partial T^\mu{}_\nu}{\partial x^1}$.
11. $\mathscr{L}_v u^a = [v,u]^a = v^b\nabla_b u^a - u^b\nabla_b v^a$. $\quad \mathscr{L}_v\omega_a = v^b\nabla_b\omega_a + \omega_b\nabla_a v^b$. $\quad \mathscr{L}_v T^a{}_b = v^c\nabla_c T^a{}_b - T^c{}_b\nabla_c v^a + T^a{}_c\nabla_b v^c$.

Proof. In the ACS $\{x^\mu\}$ of v^a, $[v,u]^\mu = (\mathrm{d}x^\mu)_a[v,u]^a = (\mathrm{d}x^\mu)_a(v^b\partial_b u^a - u^b\partial_b v^a) = v^b\partial_b(\mathrm{d}x^\mu)_a u^a = v^b\partial_b u^\mu = v(u^\mu) = \dfrac{\partial u^\mu}{\partial x^1} = (\mathscr{L}_v u)^\mu$.
12. ξ^a is **Killing** $\Longleftrightarrow$ $\mathscr{L}_\xi g_{ab} = 0 \Longleftrightarrow \nabla_{(a}\xi_{b)} = 0 \Longleftrightarrow \nabla_a\xi_b = \nabla_{[a}\xi_{b]}$, where $\nabla_a g_{bc} = 0$.
13. ξ^a is Killing, T^a is the tangent vector of a geodesic $\Rightarrow T^a\nabla_a(T^b\xi_b) = 0$. ($T^b\xi_b$ is constant along the geodesic).
14. On (M, g_{ab}), $\dim M = n$. There are at most $\frac{1}{2}n(n+1)$ independent Killing vector fields.
- $(\mathbb{R}^2, \delta_{ab})$: $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, $g_{11} = g_{22} = 1, g_{12} = 0 \Rightarrow \dfrac{\partial g_{\mu\nu}}{\partial x} = 0 \Rightarrow \left(\frac{\partial}{\partial x}\right)^a, \left(\frac{\partial}{\partial y}\right)^a$ are Killing vector fields.
- $(\mathbb{R}^3, \delta_{ab})$: $\left(\frac{\partial}{\partial x}\right)^a, \left(\frac{\partial}{\partial y}\right)^a, \left(\frac{\partial}{\partial z}\right)^a$. $-y\left(\frac{\partial}{\partial x}\right)^a + x\left(\frac{\partial}{\partial y}\right)^a, -z\left(\frac{\partial}{\partial y}\right)^a + y\left(\frac{\partial}{\partial z}\right)^a, -x\left(\frac{\partial}{\partial z}\right)^a + z\left(\frac{\partial}{\partial x}\right)^a$.
- $(\mathbb{R}^2, \eta_{ab})$: $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$. $\left(\frac{\partial}{\partial t}\right)^a, \left(\frac{\partial}{\partial x}\right)^a$. $\begin{cases} x = \psi\cosh\eta \\ t = \psi\sinh\eta \end{cases} \Rightarrow \mathrm{d}s^2 = \mathrm{d}\psi^2 - \psi^2\mathrm{d}\eta^2 \Rightarrow \left(\frac{\partial}{\partial\eta}\right)^a = t\left(\frac{\partial}{\partial x}\right)^a + x\left(\frac{\partial}{\partial t}\right)^a$.
- $(\mathbb{R}^4, \eta_{ab})$: $\left(\frac{\partial}{\partial t, x, y, z}\right)^a$. $-y\left(\frac{\partial}{\partial x}\right)^a + x\left(\frac{\partial}{\partial y}\right)^a$ (3). $\quad t\left(\frac{\partial}{\partial x}\right)^a + x\left(\frac{\partial}{\partial t}\right)^a$ (3).

15. $\phi[S]$ is hypersurface, $n_a \in V_a^*$ is the **normal covector** of $\phi[S]$ at q, if $n_a w^a = 0$, $\forall w^a \in W_q$.
- f = constant is a hypersurface, then $\nabla_a f$ is its normal covector.

16. On (M, g_{ab}), $n^a \equiv g^{ab}n_b \in V_q$ is orthogonal to all vectors on $\phi[S]$. $\quad n^a \in W_q \Longleftrightarrow n_a n^a = 0$.
17. **Induced metric** of W_q from V_q: $h_{ab}w_1^a w_2^b = g_{ab}w_1^a w_2^b$. $\quad h_{ab} = g_{ab} \mp n_a n_b$ with $n^a n_a = \pm 1$.
18. For non-null hypersurface, define $h^a{}_b \equiv g^{ac}h_{cb} = \delta^a{}_b \mp n^a n_b$. $\; h^a{}_b v^b = v^a \mp n^a(n_b v^b) \Rightarrow v^a = h^a{}_b v^b \pm n^a(n_b v^b)$.

Differential forms

1. $\omega_{a_1\cdots a_l} \in \mathscr{T}_V(0,l)$ is an ***l*-form** on V if $\boldsymbol{\omega} = \omega_{a_1\cdots a_l} = \omega_{[a_1\cdots a_l]}$. $\quad \omega_{a_1\cdots a_l} = \omega_{[a_1\cdots a_l]} \Longleftrightarrow \omega_{\mu_1\cdots\mu_l} = \omega_{[\mu_1\cdots\mu_l]}$.
2. $\boldsymbol{\omega}\wedge\boldsymbol{\mu} \equiv (\omega\wedge\mu)_{a_1\cdots a_l b_1\cdots b_m} := \dfrac{(l+m)!}{l!m!}\omega_{[a_1\cdots a_l}\mu_{b_1\cdots b_m]}$. $\wedge: \Lambda(l)\times\Lambda(m)\to\Lambda(l+m)$.
3. For 1-form, $\boldsymbol{\omega}\wedge\boldsymbol{\mu} \equiv \omega_a\wedge\mu_a = 2\omega_{[a}\mu_{b]} = \omega_a\mu_b - \omega_b\mu_a$. $\boldsymbol{\mu}\wedge\boldsymbol{\omega} = 2\mu_{[a}\omega_{b]} = \mu_a\omega_b - \mu_b\omega_a$. In general $\boldsymbol{\omega}\wedge\boldsymbol{\mu} = (-1)^{lm}\boldsymbol{\mu}\wedge\boldsymbol{\omega}$.
4. $\dim V = n \Longrightarrow \dim\Lambda(l) = \dfrac{n!}{l!(n-l)!}$ for $l \le n$. *Proof.* $\omega_{ab} = \omega_{11}(e^1)_a(e^1)_b + \omega_{12}(e^1)_a(e^2)_b + \omega_{13}(e^1)_a(e^3)_b + \omega_{21}(e^2)_a(e^1)_b + \cdots$.
$\omega_{11} = \omega_{22} = \omega_{33} = 0$, $\omega_{21} = -\omega_{12}$, $\omega_{32} = -\omega_{23}$, $\omega_{13} = -\omega_{31}$. $\omega_{ab} = \omega_{12}[(e^1)_a(e^2)_a - (e^2)_a(e^1)_a] + \omega_{23}\cdots = \omega_{12}(e^1)_a\wedge(e^2)_b + \cdots$
$\Longrightarrow \omega_{ab} \in \Lambda(2)$ can be linearly expressed by $\{(e^1)_a\wedge(e^2)_b, (e^2)_a\wedge(e^3)_b, (e^3)_a\wedge(e^1)_b\}$.
5. $\omega_{a_1\cdots a_l} = \sum_C \omega_{\mu_1\cdots\mu_l}(e^{\mu_1})_{a_1}\wedge\cdots\wedge(e^{\mu_l})_{a_l}$, $\quad \omega_{\mu_1\cdots\mu_l} = \omega_{a_1\cdots a_l}(e_{\mu_1})^{a_1}\cdots(e_{\mu_1})^{a_1}$.
- In (O,ψ) coordinate system, $\omega_{a_1\cdots a_l} = \sum_C \omega_{\mu_1\cdots\mu_l}(\mathrm{d}x^{\mu_1})_{a_1}\wedge\cdots\wedge(\mathrm{d}x^{\mu_l})_{a_l}$, $\omega_{\mu_1\cdots\mu_l} = \omega_{a_1\cdots a_l}\left(\frac{\partial}{\partial x^{\mu_1}}\right)^{a_1}\cdots\left(\frac{\partial}{\partial x^{\mu_l}}\right)^{a_l}$.
- For $l = n$, $\omega_{a_1\cdots a_n} = \omega_{1\cdots n}(\mathrm{d}x^1)_{a_1}\wedge\cdots\wedge(\mathrm{d}x^n)_{a_n}$, or $\boldsymbol{\omega} = \omega_{1\dots n}\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$.

6. **Exterior differentiation operator** $\mathrm{d}: \Lambda_M(l)\to\Lambda_M(l+1)$ is defined as $(\mathrm{d}\omega)_{ba_1\cdots a_l} := (l+1)\nabla_{[b}\omega_{a_1\cdots a_l]}$.
- $\omega_{a_1\cdots a_l} = \sum_C \omega_{\mu_1\cdots\mu_l}(\mathrm{d}x^{\mu_1})_{a_1}\wedge\cdots\wedge(\mathrm{d}x^{\mu_l})_{a_l} \Longrightarrow (\mathrm{d}\omega)_{ba_1\cdots a_l} = \sum_C(\mathrm{d}\omega_{\mu_1\cdots\mu_l})_b\wedge(\mathrm{d}x^{\mu_1})_{a_1}\cdots\wedge(\mathrm{d}x^{\mu_l})_{a_l}$.

7. $\mathrm{d}\circ\mathrm{d} = 0$. $[\mathrm{d}(\mathrm{d}\omega)]_{cba_1\cdots a_l} = (l+2)(l+1)\partial_{[c}\partial_{[b}\omega_{a_1\cdots a_l]]} = (l+2)(l+1)\partial_{[c}\partial_b\omega_{a_1\cdots a_l]} = (l+2)(l+1)\partial_{[(c}\partial_{b)}\omega_{a_1\cdots a_l]} = 0$.
8. Suppose $\boldsymbol{\omega}$ is an l-form on M. $\boldsymbol{\omega}$ is called **closed** if $\mathrm{d}\boldsymbol{\omega} = 0$. $\boldsymbol{\omega}$ is called **exact** if $\exists$ $(l-1)$-form $\boldsymbol{\mu}$ s.t. $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\mu}$.
- *Exmaple.* $M = \mathbb{R}^2$, For $X(x,y)$ and $Y(x,y)$, $\exists f(x,y)$ s.t. $\mathrm{d}f = X\mathrm{d}x + Y\mathrm{d}y \Longleftrightarrow \partial X/\partial y = \partial Y/\partial x$.

Proof. $\mathrm{d}(X\mathrm{d}x + Y\mathrm{d}y) = \mathrm{d}X\wedge\mathrm{d}x + \mathrm{d}Y\wedge\mathrm{d}y = \left(\frac{\partial X}{\partial x}\mathrm{d}x + \frac{\partial X}{\partial y}\mathrm{d}y\right)\wedge\mathrm{d}x + \left(\frac{\partial Y}{\partial x}\mathrm{d}x + \frac{\partial Y}{\partial y}\mathrm{d}y\right)\wedge\mathrm{d}y = \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y$.
9. n-dimensional manifold is called **orientable** if there exists a C^0, non-vanishing n-form field ε on it.
10. Suppose (O,ψ) is the right-handed coordinate system of n-dimensional orientable manifold M, $\boldsymbol{\omega}$ is a C^0 n-form field on $G\subset O$.
$\boldsymbol{\omega} = \omega_{1\dots n}(x^1,\cdots,x^n)\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n$, define the **integral** of $\boldsymbol{\omega}$ on G as $\int_G\boldsymbol{\omega} := \int_{\psi[G]}\omega_{1\dots n}(x^1,\cdots,x^n)\mathrm{d}x^1\cdots\mathrm{d}x^n$.
11. **Stokes theorem** in $(\mathbb{R}^3, \delta_{ab})$: $\iint_S(\nabla\times\mathbf{A})\cdot\mathrm{d}\mathbf{S} = \oint\mathbf{A}\cdot\mathrm{d}\mathbf{l}$. On manifold with boundary N: $\int_{\mathrm{i}(N)}\mathrm{d}\boldsymbol{\omega} = \int_{\partial N}\boldsymbol{\omega}$.

Example. $M=\mathbb{R}^2$, $S=\mathrm{i}(N)$, $L=\partial N$. Let $\boldsymbol{\omega}=A_a=\delta_{ab}A^b$, $A_a=A_\mu(\mathrm{d}x^\mu)_a$. $\mathrm{d}\boldsymbol{\omega}=\mathrm{d}A_\mu\wedge\mathrm{d}x^\mu=\dfrac{\partial A_\mu}{\partial x^\nu}\mathrm{d}x^\nu\wedge\mathrm{d}x^\mu=\left(\dfrac{\partial A_2}{\partial x^1}-\dfrac{\partial A_1}{\partial x^2}\right)\mathrm{d}x^1\wedge\mathrm{d}x^2$.

$\widetilde{\omega}_a=\widetilde{\omega}_1(l)(\mathrm{d}l)_a$, $\widetilde{\omega}_1(l)=\widetilde{\omega}_a\left(\dfrac{\partial}{\partial l}\right)^a=\omega_a\left(\dfrac{\partial}{\partial l}\right)^a=A_a\left(\dfrac{\partial}{\partial l}\right)^a=A_l\Longrightarrow\oint_L A_l\mathrm{d}l=\int_{\partial N}\boldsymbol{\omega}$.

12. Consider (M,g_{ab}), suppose $\varepsilon_{a_1a_2}$ is a volume element, then $\varepsilon^{a_1a_2}\equiv g^{a_1b_1}g^{a_2b_2}\varepsilon_{b_1b_2}$. If $g_{ab}\equiv\delta_{ab}/\eta_{ab}$, $\varepsilon^{a_1a_2}\varepsilon_{a_1a_2}=\pm2(\varepsilon_{12})^2$. In general, for any g_{ab}, $\varepsilon^{a_1\cdots a_n}\varepsilon_{a_1\cdots a_n}=(-1)^s n!(\varepsilon_{1\cdots n})^2$.

13. The volume element that is compatible with g_{ab} satisfies $\varepsilon^{a_1\cdots a_n}\varepsilon_{a_1\cdots a_n}=(-1)^s n!$.

- For orthonormal basis, let $\varepsilon_{1\cdots n}=\pm1$, $\varepsilon_{a_1\cdots a_n}=\pm(e^1)_{a_1}\wedge\cdots\wedge(e^n)_{a_n}$.
- $(\mathbb{R}^3,\delta_{ab})$. Let $\varepsilon=\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$. G is an open subset. $\displaystyle\int_G\varepsilon=\iiint_G\mathrm{d}x\mathrm{d}y\mathrm{d}z=V_G$.

14. The volume element that is compatible with g_{ab} satisfies $\varepsilon_{a_1\cdots a_n}=\pm\sqrt{|g|}(e^1)_{a_1}\wedge\cdots\wedge(e^n)_{a_n}$ for $\{(e_\mu)^a\}$ basis.

15. ∇_a and ε is compatible with g_{ab}, then $\nabla_b\varepsilon_{a_1\cdots a_n}=0$.

16. $\delta^{[a_1}{}_{a_1}\cdots\delta^{a_j}{}_{a_j}\delta^{a_{j+1}}{}_{b_{j+1}}\cdots\delta^{a_n]}{}_{b_n}=\dfrac{(n-j)!j!}{n!}\delta^{[a_{j+1}}{}_{b_{j+1}}\cdots\delta^{a_n]}{}_{b_n}$.

17. $\varepsilon^{a_1\cdots a_n}\varepsilon_{b_1\cdots b_n}=(-1)^s n!\delta^{[a_1}{}_{b_1}\cdots\delta^{a_n]}{}_{b_n}$. $\quad\varepsilon^{a_1\cdots a_ja_{j+1}\cdots a_n}\varepsilon_{a_1\cdots a_ja_{j+1}\cdots a_n}=(-1)^s(n-j)!j!\delta^{[a_{j+1}}{}_{b_{j+1}}\cdots\delta^{a_n]}{}_{b_n}$.

18. f is C^0 on M, define $\displaystyle\int_M f:=\int_M f\varepsilon$. For $(\mathbb{R}^2,\delta_{ab})$, $\displaystyle\int f=\int f\varepsilon=\int\omega=\iiint F(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint\hat{F}(r,\theta,\varphi)r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$.

19. N is a compact embedding submanifold in (M,g_{ab}), then $\displaystyle\int_{\mathrm{i}(N)}(\nabla_b v^b)\varepsilon=\int_{\partial N}v^b\varepsilon_{ba_1\cdots a_{n-1}}$.

20. **Gauss's Law**. $\displaystyle\iiint_V\nabla\cdot\mathbf{A}\mathrm{d}V=\oiint_S\mathbf{A}\cdot\mathbf{n}\mathrm{d}S$. On manifold with boundary N: $\displaystyle\int_{\mathrm{i}(N)}\nabla_a v^a\varepsilon=\pm\int_{\partial N}v^a n_a\hat{\varepsilon}$.

21. $\dim\Lambda_p(l)=\dfrac{n!}{l!(n-l)!}=\dim\Lambda_p(n-l)$.

22. On (M,g_{ab}), $\star:\Lambda_M(l)\to\Lambda_M(n-l)$, $\omega_{a_1\cdots a_l}\mapsto\star\omega_{a_1\cdots a_{n-l}}:=\dfrac{1}{l!}\omega^{b_1\cdots b_l}\varepsilon_{b_1\cdots b_la_1\cdots a_{n-l}}$, where $\omega^{b_1\cdots b_l}=g^{b_1c_1}\cdots g^{b_lc_l}\omega_{c_1\cdots c_l}$.

23. $f\in\mathscr{F}_M$, $\star f_{a_1\cdots a_n}=f\varepsilon_{a_1\cdots a_n}$. $\quad\star(\star f)=\star(f\varepsilon)=\dfrac{1}{n!}f\varepsilon^{b_1\cdots b_n}\varepsilon_{b_1\cdots b_n}=(-1)^s f$. $\quad$ In general, $\star\star\boldsymbol{\omega}=(-1)^{s+l(n-l)}\boldsymbol{\omega}$.

24. $\omega_{ab}\equiv A_a\wedge B_b=2A_{[a}B_{b]}\Longrightarrow\star\omega_c=\dfrac{1}{2}\omega^{ab}\varepsilon_{abc}=\varepsilon_{abc}A^{[a}B^{b]}=\varepsilon_{abc}A^aB^b$. ε_{ijk} in $\{x,y,z\}$ is orthonormal (Levi-Civita).

25. In $(\mathbb{R}^3,\delta_{ab})$, $\nabla f=\partial_a f$. $\quad\nabla\cdot\mathbf{A}=\partial_aA^a$. $\quad\nabla\times\mathbf{A}=\varepsilon^{abc}\partial_aA_b$. $\quad\nabla\cdot(\mathbf{A}\cdot\mathbf{B})=\partial_a(A^aB^b)$. $\quad\nabla^2f=\partial_a\partial^af$.

26. In $(\mathbb{R}^3,\delta_{ab})$, $\mathrm{grad}f=\mathrm{d}f$. $\quad\mathrm{curl}\,\vec{A}=\star\mathrm{d}\mathbf{A}$. $\quad\mathrm{div}\,\vec{A}=\star\mathrm{d}(\star\mathbf{A})$.

27. $\mathrm{curl}\,\vec{E}=0\Rightarrow\exists$ scalar field ϕ s.t. $\vec{E}=\mathrm{grad}\,\phi$. $\quad\mathrm{div}\,\vec{B}=0\Rightarrow\exists$ vector field $\vec{A}$ s.t. $\vec{B}=\mathrm{curl}\,\vec{A}$

Proof. $\mathrm{curl}\,\vec{E}=0\Longleftrightarrow\star(\mathrm{d}\mathbf{E})=0\Longrightarrow\mathrm{d}\mathbf{E}=0\Longrightarrow\mathbf{E}$ is closed $\Longrightarrow\mathbf{E}$ is exact $\Longrightarrow\exists\,\phi$ s.t. $\mathbf{E}=\mathrm{d}\phi\Longrightarrow\vec{E}=\mathrm{grad}\,f$.
$\mathrm{div}\,\vec{B}=0\Longrightarrow\mathrm{d}(\star\mathbf{B})=0\Longrightarrow\star\mathbf{B}$ is closed $\Rightarrow$ exact $\Longrightarrow\exists\,\mathbf{A}$ s.t. $\star\mathbf{B}=\mathrm{d}\mathbf{A}\Longrightarrow\mathbf{B}=\star\mathrm{d}\mathbf{A}$.

Special Relativity

1. In an inertial reference frame $\mathscr{R}$, the velocity of a particle at p is defined as $u\equiv\dfrac{\sqrt{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2}}{\mathrm{d}t}$. $\mathrm{d}s^2=-(1-u^2)\mathrm{d}t^2$.

2. The world line of an inertial observer is a time like geodesic, satisfying $\left(\dfrac{\partial}{\partial t}\right)^a\partial_a\left(\dfrac{\partial}{\partial t}\right)^b=0$.

3. A standard clock satisfies $\tau_2-\tau_1=\displaystyle\int_{p_1}^{p_2}\sqrt{-\mathrm{d}s^2}$.

4. Suppose $L(\tau)$ is a world line of a particle. In $\mathscr{R}$, $\mathrm{d}\tau^2=-\mathrm{d}s^2=(1-u)^2\mathrm{d}t^2\Longrightarrow\dfrac{\mathrm{d}t}{\mathrm{d}\tau}=\gamma_u=\dfrac{1}{\sqrt{1-u^2}}$. $\Delta t=\gamma\Delta\tau$.

5. $\vec{p}:=m_u\vec{u}=\gamma m\vec{u}$. $\quad f:=\mathrm{d}\vec{p}/\mathrm{d}t$.

6. $\dfrac{\mathrm{d}E_k}{\mathrm{d}t}=\vec{f}\cdot\vec{u}=\dfrac{\mathrm{d}\vec{p}}{\mathrm{d}t}\cdot\vec{u}=\vec{u}\cdot\dfrac{\mathrm{d}(m_uu)}{\mathrm{d}t}=m_u\vec{u}\cdot\dfrac{\mathrm{d}\vec{u}}{\mathrm{d}t}+\vec{u}\cdot\vec{u}\dfrac{\mathrm{d}m_u}{\mathrm{d}t}=\cdots=c^2\dfrac{\mathrm{d}m_u}{\mathrm{d}t}\Longrightarrow E_k(u)=m_uc^2-m_0c^2$. $\quad E:=\gamma m$.

7. The **4-velocity** of a particle is the tangent vector of its world line with parameter τ. $U^a:=\left(\dfrac{\partial}{\partial\tau}\right)^a$. $U^aU_a=\eta_{ab}U^bU^a=-1$.

8. Spatial vector is an element in the set $W_p=\left\{w^a\in V_p\middle|\eta_{ab}w^aZ^b=0\right\}$.

9. $U^a\equiv\left(\dfrac{\partial}{\partial\tau}\right)^a=\left(\dfrac{\partial}{\partial t}\right)^a\dfrac{\mathrm{d}t}{\mathrm{d}\tau}+\left(\dfrac{\partial}{\partial x^i}\right)^a\dfrac{\mathrm{d}x^i}{\mathrm{d}\tau}$. $U^a\mathrm{d}\tau=\left(\dfrac{\partial}{\partial\tau}\right)^a\mathrm{d}t+\left(\dfrac{\partial}{\partial x^i}\right)^a\mathrm{d}x^i=Z^a\mathrm{d}t+\left(\dfrac{\partial}{\partial x^i}\right)^a\mathrm{d}x^i$. $u^a:=\dfrac{\left(\frac{\partial}{\partial x^i}\right)^a\mathrm{d}x^i}{\mathrm{d}t}=\dfrac{\left(\frac{\partial}{\partial x^i}\right)^a\frac{\mathrm{d}x^i}{\mathrm{d}\tau}}{\mathrm{d}t/\mathrm{d}\tau}=\dfrac{h^a{}_bU^b}{\gamma}$,

where $\gamma\equiv\dfrac{\mathrm{d}t}{\mathrm{d}\tau}=-Z_aU^a$, $h_{ab}=\eta_{ab}+Z_aZ_b$. *Proof*. $-Z_aU^a=-\eta_{ab}Z^bU^a=-\eta_{\mu\nu}Z^\nu U^\mu=-\eta_{00}Z^0U^0=U^0=\frac{\mathrm{d}t}{\mathrm{d}\tau}$.

- $u^2=h_{ab}u^au^b=h_{ab}\dfrac{h^a{}_cU^ch^b{}_dU^d}{\gamma^2}=\dfrac{1}{\gamma^2}h_{cd}U^cU^d=\dfrac{1}{\gamma^2}\left(\eta_{cd}U^cU^d+Z_cZ_dU^cU^d\right)=\dfrac{1}{\gamma^2}\left(\eta_{cd}U^cU^d+\gamma^2\right)=1+\dfrac{\eta_{cd}U^cU^d}{\gamma^2}$.
- $\gamma u^a=h^a{}_bU^b=(\delta^a{}_b+Z^aZ_b)U^b=U^a-\gamma Z^a\Longrightarrow U^a=\gamma(Z^a+u^a)=\gamma'(Z'^a+u'_a)$.

10. $P^a=mU^a=EZ^a+p^a$. $\quad E=-P^aZ_a$. $\quad P^aP_a=-E^2+p^2=m^2U^aU_a=-m^2\Longrightarrow E^2=m^2+p^2$.

11. The 4-acceleration $A^a=U^b\partial_bU^a$. $A^a=0\Longleftrightarrow U^b\partial_bU^a=0$ (geodesic). $\quad A^aU_a=U_aU^b\partial_bU^a=\dfrac{1}{2}U^b\partial_b(U_aU^a)=0$.

12. The 3-acceleration on $L(\tau)$ $a^a = \frac{d^2x^i(t)}{dt^2}\left(\frac{d}{dx^i}\right)^a$. $A^0=\gamma^4\vec u\cdot\vec a$, $A^i = \gamma^2 a^i + \gamma^4(\vec u\cdot\vec a)u^i$.

13. The 4-force $F^a = U^b\partial_b P^a$ $F^i = U^b\partial_b P^a = U^b\partial_b(mU^a) = mU^b\partial_b U^a = mA^a$. $F^i=\gamma f^i$, $F^0=\gamma\vec f\cdot\vec u$, where $\vec f = \frac{d\vec p}{dt}$.

Proof. $F^\mu = F^a(dx^\mu)_a = (dx^\mu)_a U^b\partial_b P^a = U^b\partial_b[(dx^\mu)_aP^a] = U^b\partial_b P^\mu$. $F^i = U^b\partial_b p^i = \left(\frac{\partial}{\partial\tau}\right)^b\partial_b p^i = \frac{dp^i}{d\tau} = \gamma\frac{dp^i}{dt}=\gamma f^i$.

$F^0 = U^b\partial_bP^0 = \left(\frac{\partial}{\partial\tau}\right)^b\partial_b E = \frac{dE}{d\tau} = \gamma\frac{dE}{dt} = \gamma\vec f\cdot\vec u$.

14. $\vec p = \gamma m\vec u = \frac{1}{c^2}E\vec u \xrightarrow{\times 1/V}$ momentum density $= \frac{1}{c^2}$ energy density $\cdot\vec u = \frac{1}{c^2}$ energy flux.

15. The **energy-momentum tensor** T_{ab}: $T_{ab}=T_{ba}$. $\partial^aT_{ab}=0$.

16.

for instant observer $(p,(e_\mu)^a)$, $(e_0)^a=Z^a$:	$\mu = T_{ab}Z^aZ^b = T_{00}$	the energy density
	$w_i = -T_{ab}Z^a(e_i)^b = -T_{0i}$	the i-th component of the 3-momentum density
	$T_{ij} = T_{ab}(e_i)^a(e_j)^b = T^{ij}$	the ij-component of the stress tensor
	$W^a = -T^a{}_bZ^b$	the 4-momentum density

17. The 3-dimensional stress tensor $\hat T_{ab} \equiv T_{ij}(e^i)_a(e^j)_b$. $T^{ij} = \left[\hat T^{ab}(e^i)_b\right](e^j)_a \Rightarrow \hat T^{ab}$ is the 3-momentum flux density tensor.

18. $W^a = \mu Z^a + w^a$. *Proof.* $W^0 = W^a(e^0)_a = -T^a{}_bZ^b(-Z_a) = T_{ab}Z^bZ^a=\mu$. $W^i = W^a(e^i)_a = -T^a{}_bZ^b(e^i)_a = -T_{ab}Z^b(e^i)^a = w^i$.

19. $\partial_bT_{ab}=0\Rightarrow$ energy conservation.

Proof. $\partial_aW^a = \partial_a(-T^a{}_bZ^b) = -Z^b\partial^aT_{ab} - T^a{}_b\partial_aZ^b = 0 = \partial_\mu W^\mu = \partial_0W^0+\partial_iW^i = \partial_0\mu+\partial_iw^i = \frac{\partial\mu}{\partial t}+\nabla\cdot\vec w = 0$.

20. Perfect fluid $T_{ab} = \mu U_aU_b + p(\eta_{ab}+U_aU_b) = (\mu+p)U_aU_b + p\eta_{ab}$. where μ, p are scalar fields, U^a is a vector field, $U^aU_a=-1$.

21. For comoving observer $(p, U^a|_p)$, $T_{ab}(e_0)^a(e_0)^b = T_{ab}U^aU^b = (\mu+p)U_aU_bU^aU^b + p\eta_{ab}U^aU^b = \mu$, the proper energy density.

22. $\{(e_i)^a\}$ is the triad for the comoving observer, $T_{ab}(e_i)^a(e_j)^b = p\eta_{ab}(e_i)^a(e_j)^b = p\delta_{ij}\Rightarrow$ 3-dim stress tensor $\doteq \begin{bmatrix} p&0&0\\0&p&0\\0&0&p\end{bmatrix}$.

• (a) No shear stress. (b) $T_{11}=T_{22}=T_{33}\Longrightarrow$ isotropic. (c) $T_{ab}=(e_0)^a(e_i)^b=0\Rightarrow$ energy density measured by the comving observer $=0\Rightarrow$ no heat transfer.

23. $0=\partial^aT_{ab} = U_aU_b\partial^a(\mu+p) + (\mu+p)(U_a\partial^aU_b + U_b\partial^aU_a)+\partial_bp = U_bU^a\partial_a(\mu+p)+(\mu+p)(U^a\partial_aU_b+U_b\partial_aU^a)+\partial_bp$.

$\xrightarrow{\otimes U^b} 0 = -U^a\partial_a(\mu+p)+(\mu+p)(U^bU^a\partial_aU_b - \partial_aU^a)+U^b\partial_bp = U^a\partial_a\mu + (\mu+p)\partial_aU^a$. Non-rel limit: $U^a = \left(\frac{\partial}{\partial t}\right)^a + u^a$, $p\ll\mu$

$\Longrightarrow 0 = \left(\frac{\partial}{\partial t}\right)^a\partial_a\mu + u^a\partial_a\mu+\mu\partial_au^a = \frac{\partial\mu}{\partial t}+\partial_a(\mu u^a)\Longrightarrow \frac{\partial\mu}{\partial t}+\nabla\cdot(\mu\vec u)=0$ (**conservation of mass**).

$\xrightarrow{\otimes h^b{}_c=\delta^b{}_c+U^bU_c} 0 = 0+(\mu+p)(U^a\partial_aU_c+U^bU_cU^a\partial_aU_b+0)+\partial_cp+U_cU^b\partial_bp = (\mu+p)U^a\partial_aU_c+\partial_cp+U_cU^b\partial_bp$

$\xrightarrow{\otimes\left(\frac{\partial}{\partial x^i}\right)^c} 0=\mu\left[\left(\frac{\partial}{\partial t}\right)^a\partial_au_i+u^a\partial_au_i\right]+\frac{\partial p}{\partial x^i}+u_i\left[\left(\frac{\partial}{\partial t}\right)^b+u^b\right]\partial_bp = \mu\left[\frac{\partial u_i}{\partial t}+u^a\partial_au_i\right]+\frac{\partial p}{\partial x^i}+u_i\frac{\partial p}{\partial t}+u_iu^j\frac{\partial p}{\partial x^j}$

$\xrightarrow{\text{nonrel}} 0=\mu\left(\frac{\partial u_i}{\partial t}+u^a\partial_au_i\right)+\frac{\partial p}{\partial x^i}\Longrightarrow -\nabla p = \mu\left[\frac{\partial\vec u}{\partial t}+(\vec u\cdot\nabla)\vec u\right]$ (**Euler's equation**).

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24. With instantaneous obersver (p,Z^a), $E_a = F_{ab}Z^b$, $B_a = -\star F_{ab}Z^b$. $E^a=\eta^{ab}E_b$, $B^a=\eta^{ab}B_b$.

• $E_aZ^a = F_{ab}Z^aZ^b = F_{[ab]}Z^{(a}Z^{b)}=0$. $B_aZ^a = -\star F_{ab}Z^aZ^b=0$.

25. Observer: $(p,(e_\mu)^a)$. $E_i=F_{i0}$, $B_1=F_{23}$, $B_2=F_{31}$, $B_3=F_{12}$. $F_{\mu\nu}=\begin{bmatrix}0&-E_1&-E_2&-E_3\\E_1&0&B_3&-B_2\\E_2&-B_3&0&B_1\\E_3&B_2&-B_1&0\end{bmatrix}$.

• $E_i = E_a(e_i)^a = F_{ab}Z^b(e_i)^a = F_{ab}(e_0)^b(e_i)^a = F_{i0}$. $B_i = B_a(e_i)^a = -\star F_{ab}Z^b(e_i)^a = -\frac12F^{cd}\varepsilon_{abcd}(e_i)^a(e_0)^b = -\frac12F^{cd}\varepsilon_{cdi0} = -\frac12F^{\mu\nu}\varepsilon_{\mu\nu i0}$.

$B_1 = \frac12F^{\mu\nu}\varepsilon_{\mu\nu01} = \frac12(F^{23}\varepsilon_{2301}+F^{32}\varepsilon_{3201}) = F^{23}$.

26. $E_1'=E_1$, $E_2'=\gamma(E_2-vB_3)$, $E_3'=\gamma(E_3+vB_2)$. $B_1'=B_1$, $B_2'=\gamma(B_2+vE_3)$, $B_3'=\gamma(B_3-vE_2)$.

Proof. $T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}T_{\alpha\beta}$. $E_2' = T'_{20} = \frac{\partial x^2}{\partial x'^2}\frac{\partial t}{\partial t'}T_{20}+\frac{\partial x^2}{\partial x'^2}\frac{\partial x^1}{\partial t'}T_{21} = \gamma E_2-\gamma vB_3=\gamma(E_2-vB_3)$.

27. 4-current density $J^a=\rho_0U^a$. $J^a=\rho Z^a+u^a$. *Proof.* $J^a=\rho_0U^a=\rho_0\gamma(Z^a+u^a)=\rho Z^a+\rho u^a=\rho Z^a+j^a$. $\rho=-Z_aJ^a$.

• Continuity equation $\frac{\partial\rho}{\partial t}+\nabla\cdot\vec j=0$, $\partial_aJ^a=0$.

28. **Maxwell's equations**. $\partial^aF_{ab}=-4\pi J_b$, $\partial_{[a}F_{bc]}=0\Longrightarrow\nabla\cdot\vec E=4\pi\rho$, $\nabla\times\vec E=-\frac{\partial\vec B}{\partial t}$, $\nabla\cdot\vec B=0$, $\nabla\times\vec B=4\pi\vec j+\frac{\partial\vec E}{\partial t}$.

Proof. Background: $(\mathbb R^4,\eta_{ab})$ with $\partial_a,\varepsilon_{abcd}$ and (Σ_t,δ_{ab}) with $\hat\partial_a,\hat\varepsilon_{abc}$.

$\nabla\cdot\vec E = \hat\partial_aE^a = \frac{\partial E^i}{\partial x^i} = \partial_aE^a=\partial^aE_a=\partial_a(F_{ab}Z^b)=Z^b\partial^aF_{ab}=Z^b(-4\pi J_b)=4\pi\rho$.

$(\nabla\times\vec E)_c = \hat\varepsilon^{ab}{}_c\hat\partial_aE_b$. $\hat\partial_aE_b = (dx^i)_a(dx^j)_b\hat\partial_iE_j = (dx^i)_a(dx^j)_b\partial_iE_j$. $\partial_aE_b = (dx^\mu)_a(dx^j)_b\partial_\mu E_j = (dx^0)_a(dx^j)_b\partial_0E_j+(dx^i)_a(dx^j)_b\partial_iE_j$.

$h_a{}^d h_b{}^e \partial_d E_e = 0 + (\mathrm{d}x^i)_a (\mathrm{d}x^j)_b \partial_i E_j = \widehat{\partial}_a E_b$. $(\nabla \times \vec{E})_c = \widehat{\varepsilon}^{ab}{}_c \widehat{\partial}_a E_b = \widehat{\varepsilon}^{ab}{}_c h_a{}^d h_b{}^e \partial_d E_e = \widehat{\varepsilon}^{de}{}_c \partial_d E_e = \widehat{\varepsilon}^{ab}{}_c \partial_a E_b = \widehat{\varepsilon}^{ab}{}_c \partial_a (F_{be} Z^e) = Z^e \widehat{\varepsilon}^{ab}{}_c \partial_a F_{be} = -Z^e \widehat{\varepsilon}^{ab}{}_c \partial_e F_{ab} - Z^e \widehat{\varepsilon}^{ab}{}_c \partial_b F_{ea}$. $Z^e \widehat{\varepsilon}^{ab}{}_c \partial_b F_{ea} = \widehat{\varepsilon}^{ab}{}_c \partial_b (F_{ea} Z^e) = \widehat{\varepsilon}^{ab}{}_c \partial_a (F_{be} Z^e)$. $\widehat{\varepsilon}_{abc} = Z^d \varepsilon_{abcd}$.

$\Longrightarrow 2(\nabla \times \vec{E})_c = -Z^e \widehat{\varepsilon}^{ab}{}_c \partial_e F_{ab} = -Z^e Z^d \widehat{\varepsilon}_{dabc} \partial_e F^{ab} = -Z^e \partial_e (\varepsilon_{dabc} F^{ab} Z^d)$.

29. Lorentz force $\vec{f} = q(\vec{E} + \vec{u} \times \vec{B})$. Lorentz 4-force $F^a = qF^a{}_b U^b$. $qF^a{}_b U^b = U^b \partial_b P^a$.

Proof. $F^a = \gamma q F^a{}_b (Z^b + u^b) = \gamma q (E^a + F^a{}_b u^b) \Rightarrow F_a = \gamma q (E_a + F_{ab} u^b) \Rightarrow F_i = F_a (e_i)^a = \gamma q (E_i + F_{ij} u^j)$.

$(\vec{u} \times \vec{B})_c = \widehat{\varepsilon}^{ab}{}_c u_a B_b = \widehat{\varepsilon}^{ab}{}_c u_a (- \star F_{bd} Z^d) = -\widehat{\varepsilon}^{ab}{}_c u_a \left(\frac{1}{2} F^{ef} \varepsilon_{efbd} Z^d \right) = -\frac{1}{2} \widehat{\varepsilon}_{abc} u^a F_{ef} \varepsilon^{efbd} Z_d = -\frac{1}{2} Z^d \widehat{\varepsilon}_{dabc} u^a F_{ef} \varepsilon^{efbd} Z_d$.

30. $T_{ab} = \frac{1}{4\pi} \left(F_{ac} F_b{}^c - \frac{1}{4} \eta_{ab} F_{cd} F^{cd} \right) = \frac{1}{8\pi} \left(F_{ac} F_b{}^c + \star F_{ac} \star F_b{}^c \right)$.

31. $\partial_{[a} F_{bc]} = 0 \Longleftrightarrow \mathrm{d}\mathbf{F} = 0 \Longrightarrow \exists\, \mathbf{A}$ s.t. $\mathbf{F} = \mathrm{d}\mathbf{A} \Longleftrightarrow F_{ab} = (\mathrm{d}A)_{ab} = 2\partial_{[a} A_{b]} \Longrightarrow F_{ab} = \partial_a A_b - \partial_b A_a$. A_a is called the 4-potential of F_{ab}.

32. $\tilde{\mathbf{A}} = \mathbf{A} + \mathrm{d}\chi$, χ is C^2 is also a 4-potential. Lorenz gauge: $\partial^a A_a = 0$.

33. $A_a = -\phi (\mathrm{d}t)_a + a_a$.

34. $-4\pi J_b = \partial^a (\partial_a A_b - \partial_b A_a) = \partial^a \partial_a A_b - \partial_b \partial^a A_a$. Lorenz gauge: $\partial^a \partial_a A_b = -4\pi J_b$. Sourceless wave equation: $\partial^a \partial_a A_b = 0$.

35. Consider $A_b = C_b \cos\theta$ and $\partial_a C_b = 0 \Longrightarrow 0 = \partial^a \partial_a A_b = \partial^a (-C_b \sin\theta \partial_a \theta) = -C_b [\sin\theta \partial^a \partial_a \theta + \cos\theta (\partial^a \theta) \partial_a \theta] \Rightarrow \begin{cases} (\partial^a \theta) \partial_a \theta = 0 \\ \partial^a \partial_a \theta = 0 \end{cases}$.

Let $K^a = \partial^a \theta \Rightarrow K^a K_a = 0$ (null) $\Rightarrow 0 = 2K^a \partial_b K_a = 2K^a \partial_b \partial_a \theta = 2K^a \partial_a \partial_b \theta = 2K^a \partial_a K_b \Longleftrightarrow 0 = K^a \partial_a K^b \Rightarrow$ null geodesic.

Consider $\mathscr{S} = \{p \in \mathbb{R}^4 | \theta_p = \text{constant}\} \Longrightarrow K_a$ is its normal covector $\Longrightarrow K^a$ is a normal vector field $\Longrightarrow \mathscr{S}$ is null hypersurface.

Let $K^a = K^\mu \left(\frac{\partial}{\partial x^\mu} \right)^a = K^0 \left(\frac{\partial}{\partial t} \right)^a + K^i \left(\frac{\partial}{\partial x^i} \right)^a = \omega \left(\frac{\partial}{\partial t} \right)^a + k^a$. $(\mathrm{d}\theta)_a = \partial_a \theta = K_a = K_\mu (\mathrm{d}x^\mu)_a \Longrightarrow \mathrm{d}\theta = K_\mu \mathrm{d}x^\mu \xrightarrow{K_\mu = \text{ constant}} \theta = K_\mu x^\mu = K_0 t + K_i x^i = -\omega t + k_i x^i \Longrightarrow$ monochromatic plane wave $A_b = C_b \cos\left(\omega t - k_i x^i \right)$.

- $K^a = \omega Z^a + k^a \Rightarrow \omega = -K^a Z_a$. $0 = K^a K_a = (\omega Z^a + k^a)(\omega Z_a + k_a) = -\omega^2 + k^2 \Rightarrow \omega^2 = k^2$.

36. For a photon, $P^a = \hbar K^a$. $P^a = E Z^a + p^a \Rightarrow E = \hbar \omega$, $p^a = \hbar k^a$. $P^a P_a = 0 \Rightarrow E^2 = p^a p_a = p^2$.

Fundamentals of General Relativity

1. For a free particle, $0 = U^b \nabla_b P^a = m U^b \nabla_b U^a \Rightarrow$ geodesic.

2. Minimal substitution. $A^a = U^b \nabla_b U^a$, $F^a = U^b \nabla_b P^a$. $\nabla^a F_{ab} = -4\pi J_b$, $\nabla_{[a} F_{bc]} = 0$. $T_{ab} = \frac{1}{4\pi} \left(F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right)$.

3. $(\nabla_a \nabla_b - \nabla_b \nabla_a) A^a = -R_{abd}{}^a A^d = R_{bd} A^d \Longrightarrow \nabla_a \nabla_b A^a = \nabla_b \nabla_a A^a + R_{bd} A^d$.

$-4\pi J_b = \nabla^a (\nabla_a A_b - \nabla_b A_a) = \nabla^a \nabla_a A_b - \nabla^a \nabla_b A_a = \nabla^a \nabla_a A_b - \nabla_a \nabla_b A^a = \nabla^a \nabla_b A_a - \nabla_b \nabla_a A^a - R_{bd} A^d \overset{\text{L.g.}}{=\!=} \nabla^a \nabla_b A_a - R_b{}^d A_d$.

4. **Fermi-Walker deriative** $\frac{\mathrm{D_F}}{\mathrm{d}\tau} : \mathscr{F}_{G(\tau)}(k,l) \to \mathscr{F}_{G(\tau)}(k,l)$. $\frac{\mathrm{D_F} f}{\mathrm{d}\tau} = \frac{\mathrm{d}f}{\mathrm{d}\tau}$. $\frac{\mathrm{D_F} v^a}{\mathrm{d}\tau} = \frac{\mathrm{D} v^a}{\mathrm{d}\tau} + (A^a Z^b - Z^a A^b) v_b$, where $\frac{\mathrm{D} v^a}{\mathrm{d}\tau} = Z^b \nabla_b v^a$.

- If $G(\tau)$ is a geodesic, then $\frac{\mathrm{D_F} v^a}{\mathrm{d}\tau} = \frac{\mathrm{D} v^a}{\mathrm{d}\tau}$.
- $\frac{\mathrm{D_F} Z^a}{\mathrm{d}\tau} = \frac{\mathrm{D} Z^a}{\mathrm{d}\tau} + (A^a Z^b - Z^a A^b) Z_b = Z^b \nabla_b Z^a - A^a = 0$.
- $\frac{\mathrm{D_F} w^a}{\mathrm{d}\tau} = h^a{}_b \frac{\mathrm{D} w^b}{\mathrm{d}\tau}$, where w^a is a spatial vector.
- $\frac{\mathrm{D_F} g_{ab}}{\mathrm{d}\tau} = 0 \Longleftrightarrow \frac{\mathrm{D_F}(g_{ab} v^a u^b)}{\mathrm{d}\tau} = g_{ab} v^a \frac{\mathrm{D_F} u^b}{\mathrm{d}\tau} + g_{ab} u^b \frac{\mathrm{D_F} v^a}{\mathrm{d}\tau}$. *Proof.* $v^a \frac{\mathrm{D_F} u^a}{\mathrm{d}\tau} + u^a \frac{\mathrm{D_F} v^a}{\mathrm{d}\tau} = v_a \left[\frac{\mathrm{D} u^a}{\mathrm{d}\tau} + (A^a Z^b - Z^a A^b) u_b \right] + u_a \left[\frac{\mathrm{D} v^a}{\mathrm{d}\tau} + (A^a Z^b - Z\right.$
$\frac{\mathrm{D}(v_a u^a)}{\mathrm{d}\tau} + v_{(a} u_{b)} A^{[a} Z^{b]} = \frac{\mathrm{D}(v_a u^a)}{\mathrm{d}\tau} = \frac{\mathrm{D_F} g_{ab} v^a u^b}{\mathrm{d}\tau}$

5. Vector field v^a is called **Fermi-Walker transported** along $G(\tau)$ if $\frac{\mathrm{D_F} v^a}{\mathrm{d}\tau} = 0$.

- $\frac{\mathrm{D_F} v^a}{\mathrm{d}\tau} = \frac{\mathrm{D_F} u^a}{\mathrm{d}\tau} = 0 \Longrightarrow \frac{\mathrm{d}(g_{ab} v^a u^b)}{\mathrm{d}\tau} = 0 \Longrightarrow$ Inner product is conserved in Fermi transportation.

6. In Newtonian dynamics, $\vec{w}(t)$ with fixed point o is called rotational if there exists a vector s.t. $\frac{\mathrm{d}\vec{w}(t)}{\mathrm{d}t} = \vec{\omega}(t) \times \vec{w}(t)$.

7. $\frac{\mathrm{d}w^i(t)}{\mathrm{d}t} = \varepsilon^i{}_{jk} \omega^j w^k \Longrightarrow \frac{\mathrm{d}w^i(\tau)}{\mathrm{d}\tau} = \varepsilon^i{}_{jk} \omega^j w^k$. $\omega^a \xrightarrow[W_p]{h^a{}_b} \omega_a$. Let $\Omega_{ab} \equiv (\star\omega)_{ab} = \omega^c \varepsilon_{cab}$. $\varepsilon_{ijk} \omega^j w^k = -\varepsilon_{ijk} \omega^k w^j = -\Omega_{ij} w^j$

$\Longrightarrow \frac{\mathrm{d}w^i}{\mathrm{d}\tau} = -\Omega^{ij} w_j$. $\{\omega^1 = \omega^2 = 0, \omega^3 \neq 0\} \Longrightarrow \{\Omega_{23} = \Omega_{31} = 0, \Omega_{12} \neq 0\}$.

8. $G(\tau)$ is a world line in (M, g_{ab}), v^a is a vector field on $G(\tau)$. If there exists a 2-form field Ω_{ab} s.t. $\frac{\mathrm{D} v^a}{\mathrm{d}\tau} = -\Omega^{ab} v_b$, then v^a undergoes a spacetime rotation with angular velocity Ω_{ab}. $\frac{\mathrm{D} v^a}{\mathrm{d}\tau} = 0 \Rightarrow v^a$ does not rotate in spacetime.

9. v^a, u^a undergoes same spacetime rotation Ω_{ab} on $G(\tau) \Longrightarrow \frac{\mathrm{D}(v^a u_a)}{\mathrm{d}\tau} = 0$.

Proof. $\frac{\mathrm{D}(v^a u_a)}{\mathrm{d}\tau} = u_a \frac{\mathrm{D} v^a}{\mathrm{d}\tau} + v_a \frac{\mathrm{D} u^a}{\mathrm{d}\tau} = u_a (-\Omega^{ab} v_b) + v_a (-\Omega^{ab} u_b) = -2\Omega^{ab} v_{(a} u_{b)} = 0$.

10. Gauge freedom: $\Omega'_{ab}=\Omega_{ab}+\Lambda_{ab}$, $\Lambda_{ab}v_b=0 \Longrightarrow \dfrac{\mathrm{D}v^a}{\mathrm{d}\tau}=-\Omega'^{ab}v_b=-\Omega^{ab}v_b$.

11. The angular velocity 2-form of Z^a on $G(\tau)$ is $\widetilde{\Omega}_{ab}=A_a\wedge Z_b$ $(A^a\neq 0)$.

Proof. $0=\dfrac{\mathrm{D_F}Z^a}{\mathrm{d}\tau}=\dfrac{\mathrm{D}Z^a}{\mathrm{d}\tau}+(A^aZ^b-Z^aA^b)Z_b \Rightarrow \dfrac{\mathrm{D}Z^a}{\mathrm{d}\tau}=-2A^{[a}Z^{b]}Z_b=-(A^a\wedge Z^b)Z_b=-\widetilde{\Omega}^{ab}Z_b$. $\widetilde{\Omega}^{ab}\equiv A^aZ^b-Z^aA^b$.

• $\widetilde{\Omega}_{ab}=A_a\wedge Z_b$ has $\widetilde{\Omega}_{ij}=0$ $(\Omega_{0i}\neq 0)\Rightarrow$ pseudo-rotation. Ω_{ab} with $\Omega_{0i}=0\Rightarrow$ (spatial) rotation.

12. $\widetilde{\Omega}_{ab}$ is the pseudo-rotation experienced by Z^a, Ω_{ab} is the spacetime rotation of w^a. Then $\widehat{\Omega}_{ab}=\Omega_{ab}-\widetilde{\Omega}_{ab}$ is spatial rotation.

13. Spatial vector field w^a on $G(\tau)$ has no spatial rotation $\Longleftrightarrow$ It is Fermi transported along $G(\tau)$, $\dfrac{\mathrm{D_F}w^a}{\mathrm{d}\tau}=0$.

Proof. $\widehat{\Omega}_{ab}=\Omega_{ab}-\widetilde{\Omega}_{ab}\Rightarrow -\widehat{\Omega}^{ab}w_b=\dfrac{\mathrm{D}w^a}{\mathrm{d}\tau}+\widetilde{\Omega}^{ab}w_b=\dfrac{\mathrm{D_F}w^a}{\mathrm{d}\tau}$.

14. $g_{ab}\dfrac{\mathrm{D_F}w^b}{\mathrm{d}\tau}=-g_{ab}\widehat{\Omega}^{bc}w_c=-g_{ab}\varepsilon^{bcd}\omega_dw_c=-\varepsilon_{acd}\omega^dw^c=-Z^b\varepsilon_{bacd}\omega^dw^c=\varepsilon_{abcd}Z^bw^c\omega^d\Rightarrow g_{ab}\dfrac{\mathrm{D_F}w^b}{\mathrm{d}\tau}=\varepsilon_{abcd}Z^bw^c\omega^d$.
ω_a is called the spatial rotation angular velocity of spatial vecotor field w^a.

spacetime rotation	$\Omega_{ab}=\omega^c\varepsilon_{cab}$	$\dfrac{\mathrm{D}v^a}{\mathrm{d}\tau}=-\Omega^{ab}v_b$
pseudo-rotation of Z^a	$\widetilde{\Omega}_{ab}\equiv A_a\wedge Z_b$	$\dfrac{\mathrm{D}Z^a}{\mathrm{d}\tau}=-\widetilde{\Omega}^{ab}Z_b$
spatial rotation	$\widehat{\Omega}_{ab}\equiv\Omega_{ab}-\widetilde{\Omega}_{ab}$	$\dfrac{\mathrm{D_F}w^a}{\mathrm{d}\tau}=-\widehat{\Omega}^{ab}w_b$

15.